

# GRADIENT ESTIMATES IN NON-LINEAR POTENTIAL THEORY

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**ABSTRACT.** We present pointwise gradient bounds for solutions to  $p$ -Laplacean type non-homogeneous equations employing non-linear Wolff type potentials, and then prove similar bounds, via suitable caloric potentials, for solutions to parabolic equations. A method of proof entails a family of non-local Caccioppoli inequalities, together with a DeGiorgi's type fractional iteration.

## 1. THE CLASSICAL SETTING AND A ZERO ORDER ESTIMATE

In this note we describe some of the results and techniques developed in the papers [12, 22], which give a complete non-linear analog of the classical pointwise gradient estimates valid for the Poisson equation

$$(1.1) \quad -\Delta u = \mu \quad \text{in } \mathbb{R}^n,$$

where  $\mu$  is in the most general case a Radon measure with finite total mass. Moreover, the estimates we present hold for non-linear parabolic equations. At the same time our results give a somehow unexpected but natural maximal order - and parabolic - version of a by now classical result due to Kilpeläinen & Malý [17] and later extended, by mean of a different approach, by Trudinger & Wang [24]. To better frame our setting, let us recall a few basic linear results concerning the basic example (1.1) - here for simplicity considered in the whole  $\mathbb{R}^n$  - for which, due to the use of classical representation formulas, it is possible to get pointwise bounds for solutions via the use of Riesz potentials

$$(1.2) \quad I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}, \quad \beta \in (0, n]$$

such as

$$(1.3) \quad |u(x)| \leq cI_2(|\mu|)(x), \quad \text{and} \quad |Du(x)| \leq cI_1(|\mu|)(x).$$

We recall that the equivalent, localized version of the Riesz potential  $I_\beta(\mu)(x)$  is given by the linear potential

$$(1.4) \quad \mathbf{I}_\beta^\mu(x_0, R) := \int_0^R \frac{\mu(B(x_0, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}, \quad \beta \in (0, n]$$

with  $B(x_0, \varrho)$  being the open ball centered at  $x_0$ , with radius  $\varrho$ . In fact, it is not difficult to see that

$$(1.5) \quad \mathbf{I}_\beta^\mu(x_0, R) \lesssim \int_{B_R(x_0)} \frac{d\mu(y)}{|x_0-y|^{n-\beta}} = I_\beta(\mu \llcorner B(x_0, R))(x_0) \leq I_\beta(\mu)(x_0)$$

holds provided  $\mu$  is a non-negative measure. A question is now, *is it possible to give an analogue of estimates (1.3) in the case of general quasilinear equations such as for instance, the degenerate  $p$ -Laplacean equation*

$$(1.6) \quad -\operatorname{div}(|Du|^{p-2}Du) = \mu?$$

A first answer has been given in the papers [17, 24], where - for suitably defined solutions to (1.6) - the authors prove the following pointwise *zero order estimate* - i.e. for  $u$  - when  $p \leq n$ , via non-linear Wolff potentials:

$$(1.7) \quad |u(x_0)| \leq c \left( \int_{B(x_0, R)} |u|^{p-1} dx \right)^{\frac{1}{p-1}} + c \mathbf{W}_{1,p}^\mu(x_0, 2R),$$

where the constant  $c$  depends on the quantities  $n, p$ , and

$$(1.8) \quad \mathbf{W}_{\beta,p}^\mu(x_0, R) := \int_0^R \left( \frac{|\mu|(B(x_0, \varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n/p],$$

is the non-linear Wolff potential of  $\mu$ . Of course we are here using the standard notation concerning integral averages

$$\int_{B(x_0, R)} |u|^q dx := \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} |u|^q dx.$$

Estimate (1.7), which extends to a whole family of general quasi-linear equations, and which is commonly considered as a basic result in the theory of quasi-linear equations, is the natural non-linear analogue of the first linear estimate appearing in (1.3). *Here we present the non-linear analogue of the second estimate in (1.3), thereby giving a pointwise gradient estimate via non-linear potentials which upgrades (1.8) up to the gradient/maximal level.*

## 2. DEGENERATE ELLIPTIC ESTIMATES

In this section the growth exponent  $p$  will be a number such that  $p \geq 2$ , we shall therefore treat possibly degenerate elliptic equations when  $p \neq 2$ . Specifically, we shall consider general non-linear, possibly degenerate equations with  $p$ -growth of the type

$$(2.1) \quad -\operatorname{div} a(x, Du) = \mu.$$

whenever  $\mu$  is a Radon measure with finite total mass defined on  $\Omega$ ; eventually letting  $\mu(\mathbb{R}^n \setminus \Omega) = 0$ , without loss of generality we may assume that  $\mu$  is defined on the whole  $\mathbb{R}^n$ . The continuous vector field  $a: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to be  $C^1$ -regular in the gradient variable  $z$ , with  $a_z(\cdot)$  being Carathéodory regular and satisfying the following *growth, ellipticity and continuity assumptions*:

$$(2.2) \quad \begin{cases} |a(x, z)| + |a_z(x, z)|(|z|^2 + s^2)^{\frac{1}{2}} \leq L(|z|^2 + s^2)^{\frac{p-1}{2}} \\ \nu^{-1}(|z|^2 + s^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle a_z(x, z) \lambda, \lambda \rangle \\ |a(x, z) - a(x_0, z)| \leq L_1 \omega(|x - x_0|)(|z|^2 + s^2)^{\frac{p-1}{2}}, \end{cases}$$

whenever  $x, x_0 \in \Omega$  and  $z, \lambda \in \mathbb{R}^n$ , where  $0 < \nu \leq 1 \leq L$  and  $s \geq 0, L_1 \geq 1$  are fixed parameters. When  $p > 2$  we shall assume that there exists a positive  $\alpha < \min\{1, p-2\}$  such that the Hölder continuity property

$$(2.3) \quad |a_z(x, z_2) - a_z(x, z_1)| \leq L |z_2 - z_1|^\alpha (|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2-\alpha}{2}}$$

holds whenever  $z_1, z_2 \in \mathbb{R}^n$  and  $x \in \Omega$ . Here  $\omega: [0, \infty) \rightarrow [0, \infty)$  is a modulus of continuity i.e. a non-decreasing function such that  $\omega(0) = 0$  and  $\omega(\cdot) \leq 1$ . On such a function we impose a natural decay property, which is essentially optimal for the result we are going to have, and prescribes a *Dini continuous dependence of the partial map*  $x \mapsto a(x, z)/( |z| + s)^{p-1}$ :

$$(2.4) \quad \int_0^R [\omega(\varrho)]^{\frac{2}{p}} \frac{d\varrho}{\varrho} := d(R) < \infty,$$

for some  $R > 0$ . The prototype of (2.1) is - choosing  $s = 0$  and omitting the  $x$ -dependence - clearly given by the  $p$ -Laplacean equation (1.6), which satisfies (2.3) whenever  $\alpha < \min\{1, p-2\}$ . In the following, when a measure  $\mu$  actually turns out to be an  $L^1$ -function, we shall use the standard notation

$$|\mu|(A) := \int_A |\mu(x)| dx,$$

whenever  $A$  is a measurable set on which  $\mu$  is defined.

In this paper we shall present our results in the form of a priori estimates - i.e. when solutions and data are taken to be more regular than needed, for instance  $u \in C^1(\Omega)$  and  $\mu \in L^1(\Omega)$  - but they actually hold, via a standard approximation argument, for general weak and very weak solutions - i.e. distributional solutions which are not in the natural space  $W^{1,p}(\Omega)$  - to measure data problems such as, for instance

$$(2.5) \quad \begin{cases} -\operatorname{div} a(x, Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\mu$  is a general Radon measure with finite total mass, defined on  $\Omega$ . The reason for such a choice is that the approximation argument in question leads to different notions of solutions, according to the regularity/integrability properties of the right hand side  $\mu$ . We do not want to enter in such details too much, for which we refer to [12, 22], and therefore we confine ourselves to the neat a priori estimate form of the results.

For instance, in the case (2.5) with  $\mu$  being a genuinely Radon measure, in [12, 22] we consider the so called Solutions Obtained by Limit of Approximations (SOLA), which is a standard class considered when dealing with measure data problems. Such solutions are in particular unique in the case  $p = 2$ , as proved in [6, 25]. Finally, if the right hand side of (2.1) is integrable enough to deduce that  $\mu \in W^{-1,p'}(\Omega)$ , then our results apply to general weak energy solutions  $u \in W^{1,p}(\Omega)$  to (2.1).

The first result we present is now

**Theorem 2.1** (Non-linear potential gradient bound). *Let  $u \in C^1(\Omega)$ , be a weak solution to (2.1) with  $\mu \in L^1(\Omega)$ , under the assumptions (2.2). Then there exists a constant  $c \equiv c(n, p, \nu, L, L_1, \alpha) > 1$ , and a positive radius  $R_0$  depending only on  $n, p, \nu, L, L_1, \omega(\cdot), \alpha$ , such that the pointwise estimate*

$$(2.6) \quad |Du(x_0)| \leq c \left( \int_{B(x_0, R)} (|Du| + s)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} + c \mathbf{W}_{\frac{1}{p}, p}^\mu(x_0, 2R)$$

*holds whenever  $B(x_0, 2R) \subseteq \Omega$ , and  $R \leq R_0$ . Moreover, when the vector field  $a(\cdot)$  is independent of  $x$  - and in particular for the  $p$ -Laplacean operator (1.6) - estimate (2.6) holds with no restriction on  $R$ .*

The potential  $\mathbf{W}_{\frac{1}{p}, p}^\mu$  appearing in (2.6) is the natural one since its shape respects the scaling properties of the equation with respect to the estimate in question; compare with the linear estimates (1.3). When extended to general weak solutions estimate (2.6) tells us the remarkable fact that the boundedness of  $Du$  at a point  $x_0$  is independent of the solution  $u$ , and of the vector field  $a(\cdot)$  considered, but only depends on the behavior of  $|\mu|$  in a neighborhood of  $x_0$ .

A particularly interesting situation occurs in the case  $p = 2$ , when we have a pointwise potential estimate which is completely similar to the second one in (1.3), and that we think deserves a statement of its own, that is

**Theorem 2.2** (Linear potential gradient bound). *Let  $u \in C^1(\Omega)$ , be a weak solution to (2.1) with  $\mu \in L^1(\Omega)$ , under the assumptions (2.2) considered with  $p = 2$ . Then there exists a constant  $c \equiv c(n, p, \nu, L, L_1) > 0$ , and a positive radius  $R_0 \equiv R_0(n, p, \nu, L, L_1, \omega(\cdot))$  such that the pointwise estimate*

$$(2.7) \quad |Du(x_0)| \leq c \int_{B(x_0, R)} (|Du| + s) dx + c \mathbf{I}_1^{|\mu|}(x_0, 2R)$$

holds whenever  $B(x_0, 2R) \subseteq \Omega$ , and  $R \leq R_0$ . Moreover, when the vector field  $a(\cdot)$  is independent of the variable  $x$ , estimate (2.7) holds with no restriction on  $R$ .

Beside their intrinsic theoretical interest, the point in estimates (2.6)-(2.7) is that they allow to unify and recast essentially all the gradient  $L^q$ -estimates for quasilinear equations in divergence form; moreover they allow for an immediate derivation of estimates in intermediate spaces such as interpolation spaces. We refer to the recent survey [21] for an account of such estimates. Indeed, by (2.6) it is clear that the behavior of  $Du$  can be controlled by that  $\mathbf{W}_{\frac{1}{p}, p}^\mu$ , which is in turn known via the behavior of Riesz potentials. In fact, this is a consequence of the pointwise bound of the Wolff potential via the Havin-Maz'ja non linear potential [4, 14, 3], that is

$$(2.8) \quad \mathbf{W}_{\frac{1}{p}, p}^\mu(\cdot, \infty) = \int_0^\infty \left( \frac{|\mu|(B(x_0, \varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leq c I_{\frac{1}{p}} \left\{ \left[ I_{\frac{1}{p}}(|\mu|) \right]^{\frac{1}{p-1}} \right\} (x_0).$$

Ultimately, thanks to (2.8) and to the well-known properties of the Riesz potentials, we have

$$(2.9) \quad \mu \in L^q \implies \mathbf{W}_{\frac{1}{p}, p}^\mu \in L^{\frac{nq(p-1)}{n-q}} \quad q \in (1, n),$$

while Marcikiewicz spaces must be introduced for the borderline case  $q = 1$ . Inequality (2.9) immediately allows to recast the classical gradient estimates for solutions to (2.5) such as those due to Boccardo & Gall  uet [7, 8] - when  $q$  is “small” - and Iwaniec [16] and DiBenedetto & Manfredi [10] - when  $q$  is “large” - that is, for solutions to (2.5) it holds that

$$\mu \in L^q \implies Du \in L^{\frac{nq(p-1)}{n-q}} \quad q \in (1, n).$$

Moreover, since the operator  $\mu \mapsto \mathbf{W}_{\frac{1}{p}, p}^\mu$  is obviously sub-linear, using the estimates related to (2.9) and classical interpolation theorems for sub-linear operators one immediately gets estimates in refined scales of spaces such Lorentz or Orlicz spaces, recovering some estimates of Talenti [23], but directly for the gradient of solutions, rather than for solutions themselves.

Another point of Theorem 2.1 is that it allows to prove an essentially optimal Lipschitz continuity criterium with respect to the regularity of coefficients (2.4), that is

$$(2.10) \quad \mathbf{W}_{\frac{1}{p}, p}^\mu(\cdot, R) \in L^\infty(\Omega), \text{ for some } R > 0 \implies Du \in L_{\text{loc}}^\infty(\Omega, \mathbb{R}^n),$$

and moreover the local bound

$$(2.11) \quad \|Du\|_{L^\infty(B_{R/2})} \leq c \left( \int_{B(x_0, R)} (|Du| + s)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} + c \left\| \mathbf{W}_{\frac{1}{p}, p}^\mu(\cdot, R) \right\|_{L^\infty(B_R)}$$

holds whenever  $B_{2R} \subseteq \Omega$ .

We finally recall that another consequence of the classical estimate (2.8) and of (2.6) is

$$(2.12) \quad |Du(x_0)| \leq c \left( \int_{B(x_0, R)} (|Du| + s)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} + c I_{\frac{1}{p}} \left\{ \left[ I_{\frac{1}{p}}(|\mu|) \right]^{\frac{1}{p-1}} \right\} (x_0),$$

which holds whenever  $B(x_0, 2R) \subset \Omega$  satisfies the conditions imposed in Theorem 2.1. Here we recall the reader that we have previously extended  $\mu$  to the whole space  $\mathbb{R}^n$ .

### 3. PARABOLIC FIRST, AND ZERO ORDER ESTIMATES

Our aim here is not only to give a parabolic version of the elliptic estimate (2.6), but also to give a zero order estimate, that is the parabolic analog of the zero order elliptic estimate [17], the validity of which *was yet considered to be an open issue*. We consider quasilinear parabolic equations of the type

$$(3.1) \quad u_t - \operatorname{div} a(x, t, Du) = \mu,$$

in a cylindrical domain  $\Omega_T := \Omega \times (-T, 0)$ , where as in the previous section  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  and  $T > 0$ . The vector-field  $a: \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to be Carathéodory regular together with  $a_z(\cdot)$ , and indeed being  $C^1$ -regular with respect to the gradient variable  $z \in \mathbb{R}^n$ , and satisfying the following standard growth, ellipticity/parabolicity and continuity conditions:

$$(3.2) \quad \begin{cases} |a(x, t, z)| + |a_z(x, t, z)|(|z| + s) \leq L(|z| + s) \\ \nu|\lambda|^2 \leq \langle a_z(x, t, z)\lambda, \lambda \rangle \\ |a(x, t, z) - a(x_0, t, z)| \leq L_1\omega(|x - x_0|)(|z| + s) \end{cases}$$

for every choice of  $x, x_0 \in \Omega$ ,  $z, \lambda \in \mathbb{R}^n$  and  $t \in (-T, 0)$ ; here the function  $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is as in (2.2)<sub>3</sub>. Note that anyway we are assuming no continuity on the map  $t \mapsto a(\cdot, t, \cdot)$ , which is considered to be a priori only measurable. In other words we are considering the analog of assumptions (2.2) for  $p = 2$ ; the reason we are adopting this restriction is that when dealing with the evolutionary  $p$ -Laplacean operator estimates assume the usual form only when using so called “intrinsic cylinders”, according the parabolic  $p$ -Laplacean theory developed by DiBenedetto [9]. These are - unless  $p = 2$  when they reduce to the standard parabolic ones - cylinders whose size locally depends on the size of the solutions itself, therefore a formulation of the estimates via non-linear potentials - whose definition is built essentially using a standard family of balls and it is therefore “universal” - is not immediate and will be the object of future investigation. We refer to [1] for global gradient estimates.

In order to state our results we need some additional terminology. Let us recall that given points  $(x, t), (x_0, t_0) \in \mathbb{R}^{n+1}$  the standard parabolic metric is defined by

$$(3.3) \quad d_{\text{par}}((x, t), (x_0, t_0)) := \max\{|x - x_0|, \sqrt{|t - t_0|}\} \approx \sqrt{|x - x_0|^2 + |t - t_0|}$$

and the related metric balls with radius  $R$  with respect to this metric are given by cylinders  $B(x_0, R) \times (t_0 - R^2, t_0 + R^2)$ . The “caloric” Riesz potential - compare with elliptic one defined in (1.2), and with [2], for instance - is now built starting from (3.3)

$$(3.4) \quad I_\beta(\mu)((x, t)) := \int_{\mathbb{R}^{n+1}} \frac{d\mu((\tilde{x}, \tilde{t}))}{d_{\text{par}}((\tilde{x}, \tilde{t}), (x, t))^{N-\beta}}, \quad 0 < \beta \leq N := n + 2,$$

whenever  $(x, t) \in \mathbb{R}^{n+1}$ . In order to be used in estimates for parabolic equations, it is convenient to introduce its local version via the usual backward parabolic

cylinders - with “vertex” at  $(x_0, t_0)$  - that is

$$(3.5) \quad Q(x_0, t_0; R) := B(x_0, R) \times (t_0 - R^2, t_0),$$

so that we define

$$(3.6) \quad \mathbf{I}_\beta^\mu(x_0, t_0; R) := \int_0^R \frac{\mu(Q(x_0, t_0; \varrho))}{\varrho^{N-\beta}} \frac{d\varrho}{\varrho} \quad \text{where } \beta \in (0, N].$$

The main result in the parabolic case is

**Theorem 3.1** (Parabolic potential gradient bound). *Under the assumptions (3.2) and (2.4), let  $u \in C^0(-T, 0; L^2(\Omega))$  be a weak solution to (3.1) with  $\mu \in L^\infty(\Omega_T)$  and such that  $Du \in C^0(\Omega_T)$ . Then there exists a constant  $c \equiv c(n, \nu, L)$  and a radius  $R_0 \equiv R_0(n, \nu, L, L_1, \omega(\cdot))$  such that the following estimate:*

$$(3.7) \quad |Du(x_0, t_0)| \leq c \int_{Q(x_0, t_0; R)} (|Du| + s) dx dt + c \mathbf{I}_1^{|\mu|}(x_0, t_0; 2R),$$

holds whenever  $Q(x_0, t_0; 2R) \subseteq \Omega$ , and  $R \leq R_0$ . When the vector field  $a(\cdot)$  is independent of the space variable  $x$ , estimate (3.7) holds with no restriction on  $R$ .

Again, as in the elliptic case, estimate (3.7) also holds for solutions to general measure data problems as

$$(3.8) \quad \begin{cases} u_t - \operatorname{div} a(x, t, Du) = \mu & \text{in } \Omega_T \\ u = 0 & \text{on } \partial_{\text{par}} \Omega_T, \end{cases}$$

where  $\mu$  is a general Radon measure with finite mass on  $\Omega_T$ , that we shall again consider to be defined in the whole  $\mathbb{R}^{n+1}$ . In the spirit of the elliptic result (2.11) we have the following implication, which provides a boundedness criteria for the spatial gradient, under the Dini continuity assumption for the spatial coefficients stated in (2.4):

$$(3.9) \quad \mathbf{I}_1^{|\mu|}(\cdot; R) \in L^\infty(\Omega_T), \text{ for some } R > 0 \implies Du \in L_{\text{loc}}^\infty(\Omega_T, \mathbb{R}^n).$$

We conclude with the zero order potential estimate, which applies to general equations of the type (3.1) when considered with a measurable dependence upon the coefficients  $(x, t)$ . The relevant hypotheses here are the following standard growth and monotonicity properties:

$$(3.10) \quad \begin{cases} |a(x, t, z)| \leq L(|z| + s) \\ \nu |z_2 - z_1|^2 \leq \langle a(x, t, z_2) - a(x, t, z_1), z_2 - z_1 \rangle \end{cases}$$

which are assumed to hold whenever  $(x, t) \in \Omega_T$  and  $z, z_1, z_2 \in \mathbb{R}^n$ . In particular, since the pointwise bound will be derived on  $u$ , rather than on  $Du$ , we do not need any differentiability assumption on  $a(\cdot)$  with respect to the spatial gradient variable  $z$ -variable, assumptions (3.10) are clearly weaker than (3.2).

**Theorem 3.2.** *Under the assumptions (3.10), let  $u \in L^2(-T, 0; W^{1,2}(\Omega)) \cap C^0(\Omega_T)$  be a weak solution to (3.1) with  $\mu \in L^1(\Omega_T)$ . Then there exists a constant  $c$ , depending only on  $n, \nu, L, L_1$  such that the following inequality holds whenever  $Q(x_0, t_0; 2R) \subseteq \Omega$ :*

$$(3.11) \quad |u(x_0, t_0)| \leq c \int_{Q(x_0, t_0; R)} (|u| + s) dx dt + c \mathbf{I}_2^{|\mu|}(x_0, t_0; 2R) + cRs.$$

## 4. A NON-LOCAL CACCIOPPOLI'S INEQUALITY

In [12, 22] we have developed more than one approach to the proof of the point-wise gradient estimates via non-linear potentials. Here we shall present one of these, taken from [22], for the case  $p = 2$ , and for simplicity restricting to equations with no coefficients i.e. of the type

$$(4.1) \quad \operatorname{div} a(Du) = \mu.$$

We believe that such method of proof is of independent technical interest since it potentially applies to all those problems with a lack of full differentiability, as it will be clear in a few lines. Moreover, we shall see that in the case (4.1) estimate (2.7) holds component-wise; see (4.11) below. The assumptions considered for (4.1) are of course

$$(4.2) \quad \nu|\lambda|^2 \leq \langle a_z(z)\lambda, \lambda \rangle, \quad |a_z(z)| \leq L, \quad |a(0)| \leq L.$$

which hold whenever  $z, \lambda \in \mathbb{R}^n$ , where  $0 < \nu \leq L$ . The presentation of this technique is indeed one of the objectives of [22]. Aiming at the explanation of a general viewpoint, let us recall that for energy solutions  $u \in W^{1,2}(\Omega)$  to homogeneous equations of the type

$$(4.3) \quad \operatorname{div} a(Du) = 0$$

the local boundedness of the gradient is achieved by *first* differentiating the equation (4.3), proving that  $Du \in W_{\operatorname{loc}}^{1,2}(\Omega)$ , and *then* observing that  $v := D_\xi u$  solves the linear equation with measurable coefficients

$$\operatorname{div}(A(x)Dv) = 0 \quad A(x) := a_z(Du(x)).$$

At this stage the boundedness of  $D_\xi v$  follows applying an iteration method, as for instance the one devised in the pioneering work of DeGiorgi [11]. This is in turn based on the use of *Caccioppoli's inequalities on level sets*, that is, denoting

$$(w - k)_+ := \max\{w - k, 0\}, \quad (w - k)_- := \max\{k - w, 0\}$$

we have that inequalities of the type

$$(4.4) \quad \int_{B_{R/2}} |D(D_\xi u - k)_+|^2 dx \leq \frac{c}{R^2} \int_{B_{R/2}} |(D_\xi u - k)_+|^2 dx$$

and similar variants, for instance involving  $(D_\xi u - k)_-$ , hold whenever  $k \in \mathbb{R}$ . In turn, the iteration of such inequalities yields the boundedness of  $D_\xi u$ . In such an iteration, *one controls the level sets of  $D_\xi u$*  via the higher order derivatives  $D(D_\xi u - k)_+$  and Sobolev embedding theorem, building a geometric iteration in which, at every step, the gain is dictated by the Sobolev embedding exponent.

Applying such a reasoning to the case (4.1) seems to be difficult, as even in the simplest case (1.1) it is in general false that  $Du \in W^{1,1}(\Omega)$  when the right hand side  $\mu$  is just a measure, or an  $L^1$ -function. On the other hand, a result of [19] states that, although Calderón-Zygmund theory does not apply in the classical  $W^{1,1}$ -sense, when considering the borderline case when  $\mu$  is a measure or lies in  $L^1$ , it nevertheless holds *provided the right functional setting* is considered, i.e. using Fractional Sobolev spaces. Indeed, for SOLA to measure data problems as (4.1) it holds that

$$(4.5) \quad Du \in W_{\operatorname{loc}}^{1-\varepsilon,1}(\Omega, \mathbb{R}^n) \quad \text{for every } \varepsilon \in (0, 1),$$

with related explicit a priori local estimates; see [19, Theorem 1.2] for precise statements. We here recall that, for a bounded open set  $A \subset \mathbb{R}^n$  and  $k \in \mathbb{N}$ , parameters  $\alpha \in (0, 1)$  and  $q \in [1, \infty)$ , the fractional Sobolev space  $W^{\alpha,q}(A, \mathbb{R}^k)$  consists of those



measurable mappings  $w: \Omega \rightarrow \mathbb{R}^k$  such that the following Gagliardo-type norm is finite:

$$\begin{aligned} \|w\|_{W^{\alpha,q}(A)} &:= \left( \int_A |w(x)|^q dx \right)^{\frac{1}{q}} + \left( \int_A \int_A \frac{|w(x) - w(y)|^q}{|x - y|^{n+\alpha q}} dx dy \right)^{\frac{1}{q}} \\ (4.6) \quad &=: \|w\|_{L^q(A)} + [w]_{\alpha,q;A} < \infty. \end{aligned}$$

With such a notation (4.5) means that

$$(4.7) \quad [Du]_{1-\varepsilon,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x - y|^{n+1-\varepsilon}} dx dy < \infty$$

holds for every  $\varepsilon \in (0, 1)$ , and every subdomain  $\Omega' \Subset \Omega$ ; the previous quantity is intuitively the  $L^1$ -norm of the “ $(1 - \varepsilon)$ -order derivative” of  $Du$ , roughly denotable by  $D^{1-\varepsilon}Du$ . The inequality in (4.7) let us think that Caccioppoli type inequality (4.4) should be replaced by a fractional order version, and using the  $L^1$ -norm, rather than the  $L^2$ -one. Indeed we have the following theorem, that we again for simplicity state under the form of a priori estimate - i.e. assuming more regularity  $u \in W^{1,2}(\Omega)$  and  $\mu \in L^2(\Omega)$  (this can be again removed via an approximation scheme, and by considering suitable definitions of solutions). Needless to say, what it matters here is the precise form of the a priori estimate.

**Theorem 4.1** (Non-local Caccioppoli inequality). *Let  $u \in W^{1,2}(\Omega)$  be a weak solution to (4.1) with  $\mu \in L^2(\Omega)$ , under the assumptions (4.2); whenever  $\xi \in \{1, \dots, n\}$ ,  $k \geq 0$ , and whenever  $B_R \subseteq \Omega$  is a ball with radius  $R$ , the inequality*

$$(4.8) \quad [(|D_\xi u| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|D_\xi u| - k)_+ dx + \frac{cR|\mu|(B_R)}{R^\sigma},$$

*holds for every  $\sigma < 1/2$ , where the constant  $c$  depends only on  $n, \nu, L, \sigma$ .*

Comparing (4.8) and (4.4), Theorem 4.1 tells us that for quasilinear equations Caccioppoli's inequalities are a robust tool that keeps holding at intermediate derivatives/integrability levels. We do think that the idea of using non-local Caccioppoli inequalities instead of the usual ones is interesting in itself as it leads to certain types of iterations which work without fully differentiating the equation; in turn, this could apply to all those problems with a lack of full differentiability. We indeed explicitly note here that a fractional Caccioppoli inequality has been indeed derived for notwithstanding the problems has integer order. The proof of the inequality is developed in [22] and has as a starting point some techniques introduced in [18, 19].

The idea is now rather natural: inequality (4.8) serves to start an iteration in which, at each stage we control the level set of  $D_\xi u$  via the fractional derivative  $D^\sigma(D_\xi u)$  and the fractional version of Sobolev embedding theorem. We come up again with a geometric iteration whose step is in turn dictated by the fractional Sobolev embedding exponent. A point we want to emphasize, is that, as clearly inferrable from [22], inequality (4.8) contains all the information about the pointwise gradient estimate, no matter how small  $\sigma$  is taken. As a matter of fact in the following we are not using explicitly the fact that  $u$  is a solution, but rather the fact that  $D_\xi u$  satisfies (4.8). For this reason, we shall report the next result in an abstract way. Moreover, we think that the formulation below could be useful in different contexts.

**Theorem 4.2** (De Giorgi's fractional iteration). *Let  $w \in L^1(\Omega)$  be a function with the property that there exist  $\sigma \in (0, 1)$  and  $c_1 \geq 1$ , and a Radon measure  $\mu$ , such that whenever  $B_R \subseteq \Omega$  is a ball with radius  $R$  and  $k \geq 0$ , the inequality*

$$(4.9) \quad [(|w| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c_1}{R^\sigma} \int_{B_R} (|w| - k)_+ dx + \frac{c_1 R |\mu|(B_R)}{R^\sigma},$$



holds. Then the following estimate:

$$(4.10) \quad |w(x_0)| \leq c \int_{B(x_0, R)} |w| dx + c \mathbf{I}_1^{|\mu|}(x_0, 2R)$$

holds whenever  $B(x_0, 2R) \subset \Omega$ , where the constant  $c$  depends on  $c_1, n, \sigma$ .

The dependence of the constant  $c$  appearing in (4.10) is not surprisingly as follows:

$$\lim_{\sigma \rightarrow 0} c = \infty \quad \text{and} \quad \lim_{c_1 \rightarrow \infty} c = \infty.$$

Now we just have to conclude merging the last two theorems. Indeed, by Theorem 4.1 we have assumption (4.9) from Theorem 4.2 satisfied by  $w \equiv D_\xi u$ . In turn, applying Theorem 4.2 with such a choice of  $w$  we conclude with the desired pointwise gradient bound

$$(4.11) \quad |D_\xi u(x_0)| \leq c \int_{B(x_0, R)} |D_\xi u| dx + c \mathbf{I}_1^{|\mu|}(x_0, 2R).$$

The last estimate clearly implies (2.7), being actually stronger since it holds for each single component of the gradient.

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